

# Logarithmic Curvature and Torsion Graphs

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**Abstract.** This paper introduces logarithmic curvature and torsion graphs for analyzing planar and space curves. We present a method for drawing these graphs from any differentiable parametric curves and clarify the characteristics of these graphs. We show several examples of these graphs drawn from planar and 3D Bézier curves. From the graphs, we can see some interesting properties of curves that cannot be derived from the curvature or torsion plots.

## 1 Introduction

To design high quality shapes, CAD systems are usually equipped with tools for inspecting the geometric quality of curves and surfaces. For curves, the curvature plot, which is the plot of curvature versus arc length or versus the parameter, is commonly used[1]. For surfaces, the color coding of Gaussian curvature or mean curvature, the reflection lines[8], or isophotes[10] are used. This paper proposes a new method for identifying the shape information of planar and space curves. For planar curves, we propose to use logarithmic curvature graphs (LCGs). For space curves, we propose to use logarithmic curvature graphs and logarithmic torsion graphs (LTGs). Note that the term “logarithmic curvature” has been used by Farouki et al. in [3]. We use the term in an entirely different context.

A logarithmic curvature graph is a graph whose horizontal and vertical axes are  $\log \rho$  and  $\log \left( \rho \frac{ds}{d\rho} \right)$ [9, 11], where  $\rho$  and  $s$  are the radius of curvature and the arc length, respectively. An LCG is the analytical representation of a logarithmic curvature histogram(LCH) originally proposed by Harada[6, 7]. Harada has used LCHs to analyze curvature variation properties of curves in existing aesthetic objects and discovered that LCHs (i.e. LCGs) of many such curves become linear. The overall shapes of curves whose LCG is linear has been clarified and a method for drawing a curve segment has been proposed in [11]. If the LCG of a curve is linear, its slope  $\alpha$  is known as the type of the curve[6]. If  $\alpha = -1, 0, 1$  or  $2$ , the curve is known to be the Clothoid, the Nielsen’s spiral[5], the logarithmic spiral, or the circle involute[6, 11]. Conversely, if we could draw the LCG of a planar curve and found that the LCG is almost linear with slope  $\alpha$ , we can find the type of the curve. If  $\alpha$  is close to  $-1$ , it means that the curve is close to the

Clothoid. To identify the shape information of space curves, we need to append LTGs onto LCGs. The role of torsion variation is not clearly known for aesthetic space curves. However, for a certain class of space curves, if we could know both the LCGs and LTGs are linear, we could get the shape information of the curve. LCGs and LTGs can be used to find the shape information of curves.

In this paper, we propose LCGs and LTGs to be drawn from any differential parametric curves, such as Bézier curves, or NURBS curves. We clarify the characteristics of LCGs and LTGs and show several examples of these graphs drawn from Bézier curves. LCGs and LTGs do not replace curvature and torsion plots. LCGs and LTGs offer the information related to the curvature and torsion variations that the curvature and torsion plots do not offer. For planar curves, LCGs are already introduced in [12, 13]. In this paper, however, we draw LCGs and LTGs from space curves and derive more detailed characteristics. Moreover, using the LCGs, we show that we can find an interesting property of polynomial curves near the inflection point. Using LCGs and LTGs, we show that the slopes of LCGs and LTGs of typical 3D class A Bézier curves[2] get closer to 1 as the degree gets higher.

## 2 Logarithmic Curvature Graphs and their Characteristics

### 2.1 Logarithmic Curvature Graphs

LCGs are defined to be graphs whose horizontal and vertical axes are  $\log \rho$  and  $\log \left( \rho \left| \frac{ds}{d\rho} \right| \right)$ , respectively. See Fig.1(a). Comparing the previous definition of LCGs, which was derived from the linearity of LCG[9, 11]

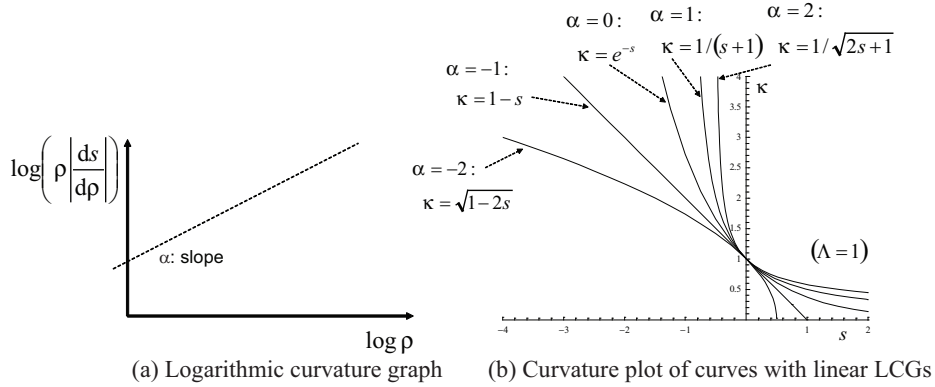
$$\log \left( \rho \frac{ds}{d\rho} \right) = \alpha \log \rho + c , \quad (1)$$

the absolute value is taken for  $\frac{ds}{d\rho}$ . This is because, in the previous definition,  $\rho$  is assumed to be monotonically increasing with respect to  $s$ . The effect of taking the absolute value can be considered as reparameterizing the curve so that the radius of curvature becomes monotonically increasing with respect to the arc length.

Let  $\mathbf{x}(t)$  be a differentiable curve with parameter  $t$ . We represent the first, second and third derivatives of  $\mathbf{x}(t)$  with respect to  $t$  by  $\dot{\mathbf{x}}$ ,  $\ddot{\mathbf{x}}$  and  $\dddot{\mathbf{x}}$ , respectively. To draw the LCG, we need to compute both  $\log \rho$  and  $\log \left( \rho \left| \frac{ds}{d\rho} \right| \right)$ . We can easily compute  $\log \rho$ . Let  $\kappa$  be the curvature. We can compute  $\log \left( \rho \left| \frac{ds}{d\rho} \right| \right)$  using the following equations:

$$\frac{d\rho}{ds} = -\frac{1}{\kappa^2} \frac{d\kappa}{ds} , \quad (2)$$

$$\frac{d\kappa}{ds} = \frac{(\dot{\mathbf{x}} \wedge \ddot{\mathbf{x}}) \cdot (\dot{\mathbf{x}} \wedge \ddot{\mathbf{x}})(\dot{\mathbf{x}} \cdot \dot{\mathbf{x}}) - 3|\dot{\mathbf{x}} \wedge \ddot{\mathbf{x}}|^2(\dot{\mathbf{x}} \cdot \ddot{\mathbf{x}})}{|\dot{\mathbf{x}}|^6 |\dot{\mathbf{x}} \wedge \ddot{\mathbf{x}}|} . \quad (3)$$



**Fig. 1.** Logarithmic curvature graph and the curvature plot of curves with linear LCGs

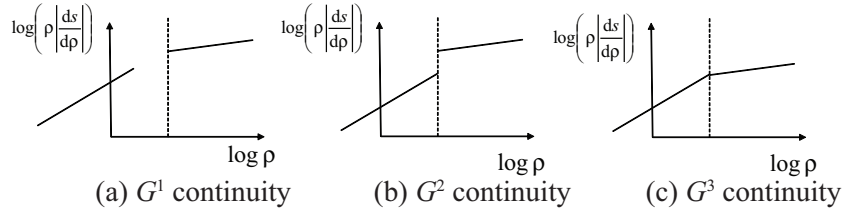
Note that (3) works for both planar and space curves. For planar curves, however, it is more efficient to use the simplified version of  $\frac{d\kappa}{ds}$  [1, 11].

At the point of  $\frac{d\rho}{ds} = 0$  (or  $\frac{d\kappa}{ds} = 0$ ), the vertical value of the LCG is not defined. As the curve gets close to the point of  $\frac{d\rho}{ds} = 0$ , the vertical value gets close to infinity since  $\rho$  and  $|\frac{ds}{d\rho}|$  get close to a constant and infinity, respectively. Therefore, when we draw the LCG of a curve with not monotonically varying curvature, the vertical values of the LCG contain the same number of infinite values as the number of curvature extrema. Both the vertical and the horizontal values of the LCG is not well-defined at the point of  $\rho = 0$  or  $\rho = \infty$ . In [11], we have clarified that a curve with linear LCG whose slope is greater than or equal to zero has an inflection point at infinity. A curve with linear LCG whose slope is smaller than 0 has an inflection point (not at infinity). Corresponding to these properties, if the radius of curvature of a curve approaches infinity as the arc length approaches infinity, both the horizontal and vertical values of the LCG approaches  $\infty$ . If the LCG is linear, this means that the slope of the LCG is greater than or equal to 0. If the radius of curvature of a curve becomes infinite not at infinity, the horizontal value of the LCG becomes  $\infty$  whereas the vertical value becomes  $-\infty$  at that point. Again, if the LCG is linear, this means that the slope of the LCG is smaller than 0.

Modifying (1), we get the curvature function with respect to the arc length [11]:

$$\kappa = \begin{cases} e^{-\Lambda s} & \text{if } \alpha = 0 \\ (\Lambda \alpha s + 1)^{-\frac{1}{\alpha}} & \text{otherwise} \end{cases} \quad (4)$$

where  $\Lambda = e^{-c}$ . See Fig.1(b). If the LCG is (approximately) linear, the curvature function can be (approximately) represented by (4).



**Fig. 2.** LCGs of curves with  $G^1$ ,  $G^2$  and  $G^3$  continuity

## 2.2 Continuity of Curves and Continuity of LCGs

We consider the continuity of a curve and the continuity of its LCG. If a curve is  $G^1$  (or less than  $G^1$ ) continuous at some point of the curve, both the horizontal and vertical values of the LCG becomes different at the point. If a curve is  $G^2$  continuous at some point, the horizontal values of the LCG becomes the same. But the vertical values become different at the point. If a curve is  $G^3$  continuous at some point of the curve, the LCG of the curve becomes positionally( $G^0$ ) continuous since both the horizontal values and vertical values take the same values. See Fig.2 for the continuity of curves and the continuity of LCGs. Note that the positional continuity of the LCG does not necessarily mean that the curve is  $G^3$  continuous. Suppose there are two curve segments with  $G^3$  continuity. Translating or rotating one of the two segments does not change the LCG since it does not affect the differential properties of curves.

## 2.3 Formulation by Curvature

LCGs in the above sections are formulated by the radius of curvature  $\rho$ . LCGs can also be formulated by curvature  $\kappa$ . Except for this section, we say LCGs to mean LCGs formulated by the radius of curvature.

We simply consider replacing  $\rho$  of LCGs formulated by the radius of curvature by  $\kappa$ . Then the horizontal and vertical values of LCGs formulated by curvature are  $\log \kappa$  and  $\log(\kappa \left| \frac{ds}{d\kappa} \right|)$ , respectively. Since

$$\log \rho = -\log \kappa, \quad (5)$$

horizontal values of LCGs formulated by the radius of curvature and LCGs formulated by curvature have the same absolute value but opposite signs.

Using (2) and  $\kappa = \frac{1}{\rho}$ ,

$$\rho \frac{ds}{d\rho} = -\kappa \frac{ds}{d\kappa}. \quad (6)$$

Since  $\rho$  and  $\kappa$  are both positive, taking the absolute values of  $\frac{ds}{d\rho}$  and  $\frac{ds}{d\kappa}$ , we get

$$\rho \left| \frac{ds}{d\rho} \right| = \kappa \left| \frac{ds}{d\kappa} \right|. \quad (7)$$

Therefore, the vertical values of LCGs formulated by the radius of curvature and the vertical values of LCGs formulated by curvature are the same. Thus the LCG of a curve formulated by the radius of curvature is symmetric to the LCG formulated by curvature with respect to the vertical axis of  $\log \rho = 0$ .

### 3 Logarithmic Torsion Graphs and their Characteristics

#### 3.1 Logarithmic Torsion Graphs

We define logarithmic torsion graphs by replacing  $\rho$  of LCGs by the radius of torsion  $\mu$ . The linearity of the LTG can be represented by

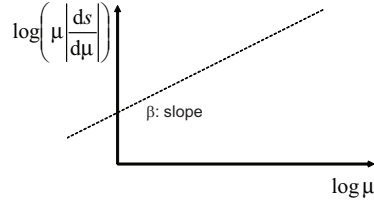
$$\log \left( \mu \frac{ds}{d\mu} \right) = \beta \log \mu + d , \quad (8)$$

where  $\beta$  is the slope of the straight line and  $d$  is a constant. See Fig.3 for the LTG. Similar properties as in LCGs hold for LTGs. Thus we try to abbreviate the explanation as much as possible.

Similarly as in Section 2.1, we represent a curve by  $\mathbf{x}(t)$  and denote the fourth derivative of  $\mathbf{x}(t)$  with respect to  $t$  by  $\ddot{\ddot{\mathbf{x}}}$ . Let  $\tau$  be the torsion. Then the LTG of a curve can be drawn using the following equations:

$$\frac{d\mu}{ds} = -\frac{1}{\tau^2} \frac{d\tau}{ds} , \quad (9)$$

$$\frac{d\tau}{ds} = \frac{(\dot{\mathbf{x}} \wedge \ddot{\mathbf{x}}) \cdot \ddot{\ddot{\mathbf{x}}} |\dot{\mathbf{x}} \wedge \ddot{\mathbf{x}}|^2 - 2((\dot{\mathbf{x}} \wedge \ddot{\mathbf{x}}) \cdot \ddot{\mathbf{x}}) (\dot{\mathbf{x}} \wedge \ddot{\mathbf{x}}) \cdot (\dot{\mathbf{x}} \wedge \ddot{\mathbf{x}})}{|\dot{\mathbf{x}} \wedge \ddot{\mathbf{x}}|^4 |\dot{\mathbf{x}} \cdot \dot{\mathbf{x}}|^{\frac{1}{2}}} . \quad (10)$$



**Fig. 3.** Logarithmic Torsion Graph

At the point of  $\frac{d\mu}{ds} = 0$  (or  $\frac{d\tau}{ds} = 0$ ), the vertical value of LTG becomes infinite. Therefore at the torsion extremum, the vertical value of the LTG becomes infinite. At the point of  $\mu = 0$  or  $\mu = \infty$ , both the horizontal and vertical values of the LTG are not defined. Similarly as in LCG, as the curve approaches the point at  $\mu = \infty$  in infinite distance, both the horizontal and vertical values of LTG approaches  $\infty$ . If the point at  $\mu = \infty$  is in a finite distance, the horizontal

value of LTG approaches  $\infty$  whereas the vertical value approaches  $\infty$ , as the curve approaches the point at  $\mu = \infty$ .

Modifying (8), we can obtain the torsion function with respect to the arc length:

$$\tau = \begin{cases} e^{-\Omega s} & \text{if } \beta = 0 \\ (\Omega\beta s + 1)^{-\frac{1}{\beta}} & \text{otherwise} \end{cases} \quad (11)$$

where  $\Omega = e^{-d}$ . The (approximate) linearity of LTG means that the torsion function is (approximately) represented by (11).

### 3.2 Continuity of Curves and Continuity of LTGs

The torsion is related to the third derivative of a curve. See Fig.4 to see the relationship between the continuity of curves and the continuity of the LTG.

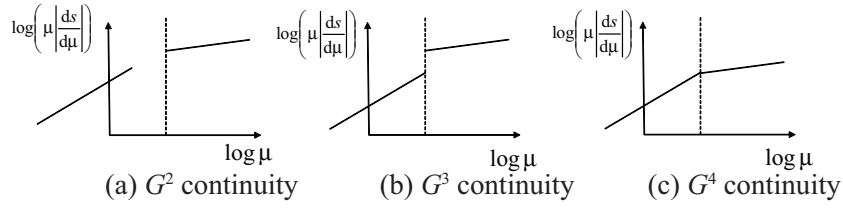


Fig. 4. LTGs of curves with  $G^2$ ,  $G^3$  and  $G^4$  continuity

### 3.3 Formulation by Torsion

Since

$$\log \mu = -\log \tau, \quad \log \mu \left| \frac{ds}{d\mu} \right| = \log \tau \left| \frac{ds}{d\tau} \right|, \quad (12)$$

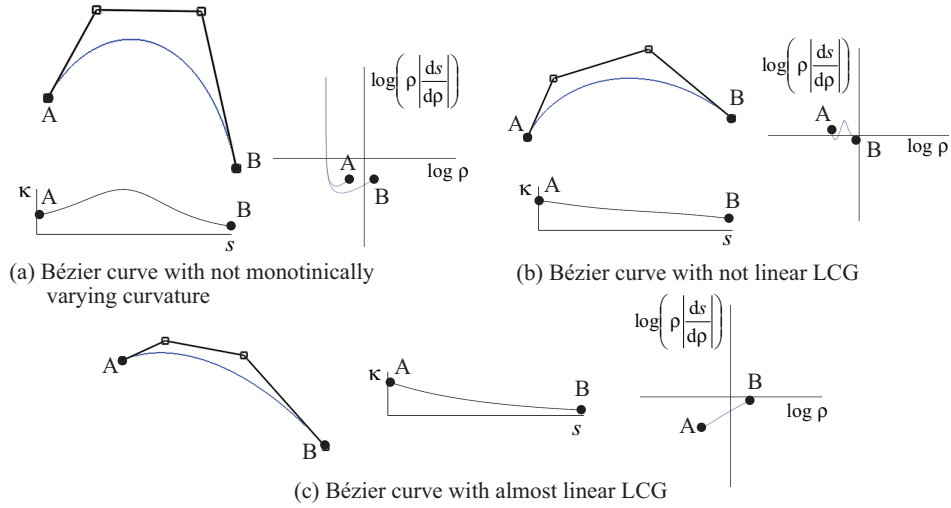
the LTG formulated by the torsion is symmetric to the LTG formulated by the radius of torsion with respect to the axis  $\log \mu = 0$ .

## 4 Examples of LCGs and LTGs

### 4.1 Planar Bézier Curves

Fig.5 shows the curve, the curvature plot, and the LCGs of three planar Bézier curves. Fig.5(a) is a Bézier curve with not monotonically varying curvature. There is a point in the LCG where the vertical value becomes infinite since there is a curvature extremum in the curve. Fig.5(b) is a Bézier curve with monotonically varying curvature. Although the curvature is monotonically varying, its LCG is not linear. Therefore this Bézier curve is not very aesthetically pleasing

from the viewpoint of Harada's analysis[6,7]. Fig.5(c) is a Bézier curve with almost linear LCG. The slope of the LCG implies the type( $\alpha$ ) of log-aesthetic curves.



**Fig. 5.** Curvature plots and LCGs of various Bézier curves

Using LCGs, we can find an interesting property of polynomial curves. Fig.6 (a) and (b) are cubic and quintic Bézier curves with inflection points. As the curve approaches the inflection point (in a finite distance), the horizontal value ( $\log \rho$ ) of the LCG approaches  $\infty$  whereas the vertical value ( $\log \left( \rho \left| \frac{ds}{d\rho} \right| \right)$ ) approaches  $-\infty$ . In both of the LCGs, we can see that the slopes approach  $-1$  as the curve approaches the inflection point. This indicates that in the vicinity of the inflection point, the curve can be approximated by the Clothoid (the log-aesthetic curve with  $\alpha = -1$ ). We found that in any polynomial Bézier curves of any degree, the slope of the LCGs near the inflection point is  $-1$  except when the control points are placed in specific positions. Fig.6(c) and (d) show the cases where the slopes of the LCGs are not  $-1$  near the inflection points. The slopes of the LCGs near the inflection point of Fig.6(c) and (d) are  $\frac{1}{2}$  and  $\frac{1}{4}$ . This indicates if the control points of a Bézier curve of degree  $n$  are placed in a specific position as shown in Fig.6 (c) or (d), the curve can be approximated, near the inflection point, by log-aesthetic curves with  $\alpha = -\frac{1}{n-2}$ . Note that LCGs do not give proofs for the properties of curves. They provide the shape information of curves for further investigation.

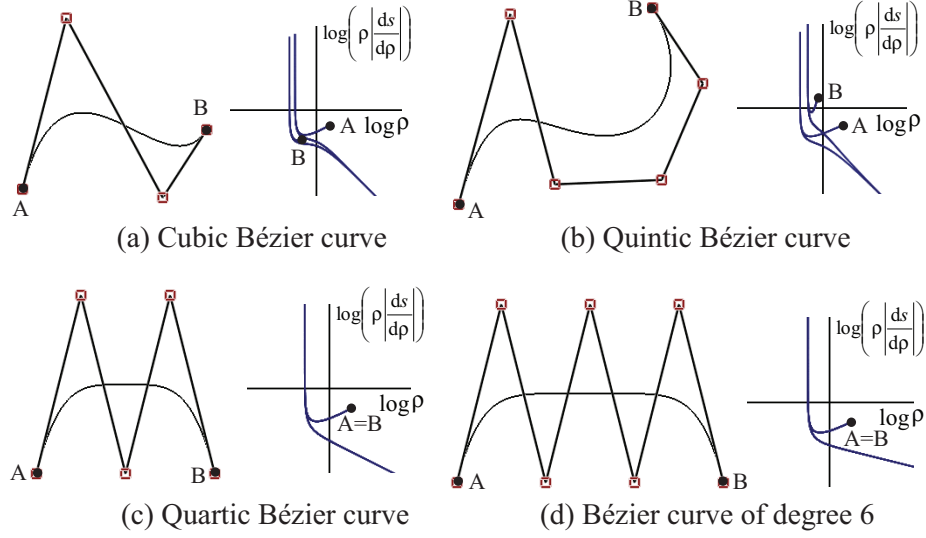


Fig. 6. LCGs of Bézier curves with inflection points

#### 4.2 3D Bézier curves

As an application of LCGs and LTGs, we show that these graphs can be used to find the shape information of space curves. We found that the LCGs and LTGs of typical 3D class A Bézier curves[2] get closer to be linear with slopes both 1, as the degree gets higher. This means that typical 3D class A Bézier curves get closer to 3D logarithmic spirals as the degree gets higher.

Let  $n$  and  $\mathbf{b}_i (0 \leq i \leq n)$  be the degree and the control points of a Bézier curve, respectively. Let  $\Delta \mathbf{b}_i = \mathbf{b}_{i+1} - \mathbf{b}_i (0 \leq i < n)$  be the forward difference of control point vectors. Typical 3D class A Bézier curves are curves with monotonically varying curvature and torsion having the relationship

$$\Delta \mathbf{b}_i = M^i \Delta \mathbf{b}_0 \quad (1 \leq i < n) \quad (13)$$

where  $M$  is a matrix composed of a rotation about an axis and a scaling. Note that (13) does not guarantee the curvature and torsion are monotonically varying. A method for drawing typical 3D class A Bézier curves by specifying two endpoints and their tangents is proposed in [4]. Drawing the LCGs and LTGs for typical 3D class A Bézier curves, we found that both LCGs and LTGs get closer to be linear with slopes 1 when the degree gets higher. Fig. 7 is an example where the degree is changed with the same endpoint constraints. In the figure, the slopes are computed by the least square method and "var." means the variance. A curve whose slopes of the LCG and LTG are both 1 is the 3D extension of a logarithmic spiral:

$$L(t) = (e^{at} \cos t, e^{at} \sin t, be^t), \quad (14)$$



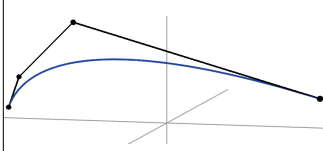
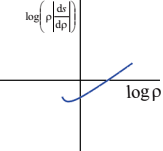
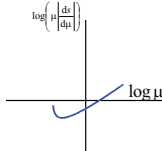
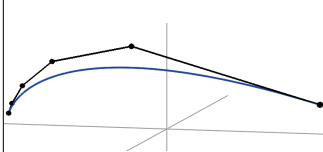
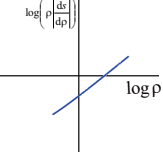
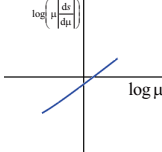
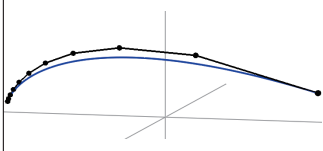
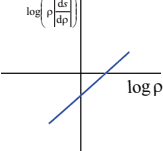
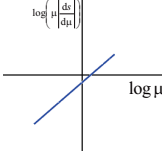
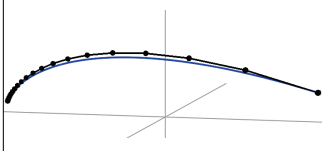
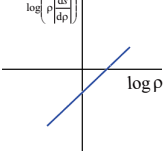
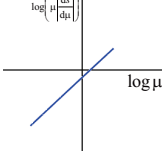
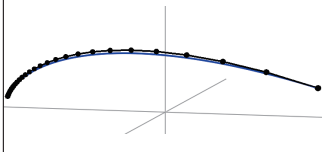
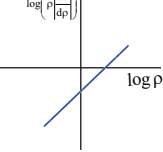
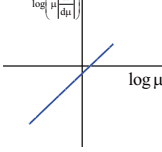
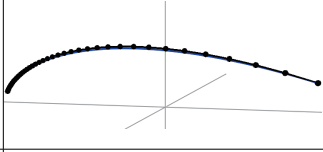
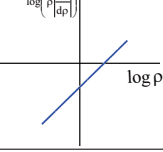
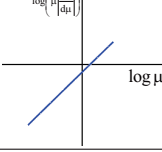
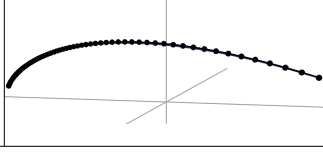
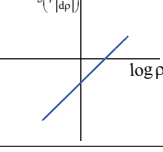
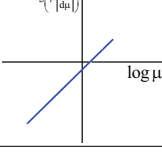
deg.	curves	LCG	slope	LTG	slope
			var		var
3			0.574		0.453
			12.15		37.12
5			0.780		0.732
			0.425		0.459
10			0.892		0.871
			0.096		0.118
20			0.945		0.936
			0.026		0.054
30			0.963		0.957
			0.003		0.011
50			0.978		0.974
			$3.0 \cdot 10^{-4}$		0.001
99			0.990		0.987
			$7.2 \cdot 10^{-5}$		$1.7 \cdot 10^{-4}$

Fig. 7. LCGs and LTGs of typical 3D class A Bézier curves

where  $a$  and  $b$  are constants. LCGs and LTGs do not give a proof that typical 3D class A Bézier curves are approximations to the 3D extensions of a logarithmic spiral. However, they give the shape information concerning the curves, which the curvature and torsion plots do not offer.

## 5 Conclusions

We have proposed logarithmic curvature graphs(LCGs) and logarithmic torsion graphs(LTGs) for analyzing the characteristics and identifying the shape information of any differentiable parametric curves. We have clarified the characteristics of these graphs and showed several examples of the graphs drawn from planar and space Bézier curves. Using LCGs and LTGs, we showed that these graphs can identify the shape information of curves, which the curvature and torsion plots do not offer. Future work includes the fairing and segmentation of curves based on the linearity of these graphs, and most importantly, the extension of logarithmic graphs to surfaces.

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